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Ricci Solitons in β-Kenmotsu Manifolds

Rajesh Kumar[1]

Abstract. The object of the present paper is to study Ricci soliton in β-Kenmotsu manifolds. Here it is proved that a symmetric parallel second order covariant tensor in a β-Kenmotsu manifold is a constant multiple of the metric tensor. Using this result, it is shown that if $(\mathcal{L}_V g + 2S)$ is ∇-parallel where V is a given vector field, then the structure (g, V, λ) yields a Ricci soliton. Further, by virtue of this result, we found the conditions of Ricci soliton in β-Kenmotsu manifold to be shrinking, steady and expending respectively. Next, Ricci soliton for 3-dimensional β-Kenmotsu manifold are discussed with an example.

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1 Introduction

The Ricci flow is an intrinsic geometric flow which was introduced by Hamilton in 1982 ([25], [26]). Ricci flow on a smooth, compact and without boundary Riemannian manifold M equipped with a Riemannian metric g satisfies the following geometric evolution equation

$$\frac{\partial g}{\partial t} = -2S, \tag{1.1}$$

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where S is Ricci curvature tensor depending on g.

Hamilton himself and many other researchers like Cao [16], Yau [31], Chow and others [5], Perelman ([14],[15]), Morgan and Tian [19] developed the theory of Ricci flow. On the other hand Hamilton [27] introduced a more general notion of Ricci soliton in the context of metric paracontact geometry. More precisely, Ricci soliton is the natural generalization of Einstein metric and is defined on a Riemannian manifold.

In a Riemannian manifold (M, g), g is called a Ricci soliton if

$$(\mathcal{L}_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0 \tag{1.2}$$

for any vector fields X, Y and V on M, where $\mathcal{L}_V$ denote the Lie derivative operator along the vector field V, S is the Ricci tensor and λ is a constant. The metric satisfying (1.2) are very interesting in the field of Physics and are often referred as quasi-Einstein ([32],[33],[12]).

The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ respectively.

In the paracontact geometry, Ricci soliton firstly was studied by Calvaruso and Perrone [12]. Recently, Bejan and Crasmareanu [8] studied Ricci solitons on 3-dimensional normal paracontact manifold.

It is know that [22] if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other then a constant multiple of the metric tensor, then it is reducible. The necessary and sufficient condition for the existence of such tensor was given by Levy [18].

The generalization of Levy's results is given by Sharma ([28],[29]). He shown that a second order parallel (not necessarily symmetric and non-singular) tensor on an n-dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. He also proved that there is no non-zero parallel 2-form in a Sasakian manifold. Das [21] studied second order parallel tensor on an almost contact metric manifold and found that on an α-K-contact manifold (α being non-zero real constant) a second order symmetric parallel tensor is a constant multiple of the associative positive definite Riemannian metric tensor. It is also proved that in an α-Sasakian manifold there is no non-zero parallel 2-form. The study of Ricci solitons in K-contact manifolds was started by Sharma [30] and in the continuation of this Ghosh, Sharma and Cho [2] studied gradient Ricci soliton of a non-Sasakian (κ, μ)-contact manifold. Generally in a P-Sasakian manifold the structure vector field ξ is not killing, that is $(\mathcal{L}_V g) \neq 0$ but in K-contact manifold ξ is a killing vector field, that is $(\mathcal{L}_V g) = 0$. Recently in [34], De have studied Ricci soliton in P-Sasakian manifolds. Barua and De [4] have studied Ricci solitons in Riemannian manifolds. Since then, several other studied

Ricci soliton in various contact manifolds: Eisenhart problem to Ricci soliton in f-Kenmotsu manifold [6], Eta-Ricci solitons on para-Kenmotsu manifolds [3], on contact and Lorentzian manifolds ([6],[9],[28]), on Sasakian manifold ([1],[7]), on α-Sasakian [13], on Kenmotsu manifold [10], etc.
Motivated by the above studies, in this paper we treat Ricci soliton in β-Kenmotsu manifolds. The paper is structured as follows. After introduction, section 2 is a brief review of β-Kenmotsu manifold. Section 3 is devoted to the study of parallel symmetric second order tensor in β-Kenmotsu manifolds and Ricci soliton in β-Kenmotsu manifolds. So we obtain a relation between symmetric parallel second order covariant tensor and metric tensor in β-Kenmotsu manifold. In the second problem of this section we studied the necessary and sufficient condition of a Ricci semi-symmetric β-Kenmotsu manifold to be an Einstein manifold. We also analyzed the behavior of Ricci soliton in an n-dimension β-Kenmotsu manifold and η-Einstein manifolds. Section 4 is devoted to study Ricci soliton in 3-dimensional β-Kenmotsu manifold with an example.

2 Preliminaries

An n-dimensional differential smooth manifold M is said to be almost contact metric manifold [11] if it admits a (1,1) tensor field φ, a contravariant vector field ξ, a 1-form η and a Riemannian metric g which satisfy

$$\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad g(X,\xi) = \eta(X), \tag{2.2}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M(\varphi,\xi,\eta,g)$ is said to be β-Kenmotsu manifold [20] if

$$(\nabla_X\varphi)(Y) = \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X]. \tag{2.3}$$

From (2.3), we have

$$\nabla_X\xi = \beta[X - \eta(X)\xi], \tag{2.4}$$

where $\beta \in C^\infty(M)$ and ∇ denote the Riemannian connection of g. If $\beta = 1$ then β-Kenmotsu manifold is called Kenmotsu manifold and if β is constant then it is called homothetic Kenmotsu manifold.

In an n-dimensional β-Kenmotsu manifold, the following relations hold [20]:

$$R(X,Y)\xi = -\beta^2[\eta(Y)X - \eta(X)Y] + (X\beta)[Y - \eta(Y)\xi] - (Y\beta)[X - \eta(X)\xi], \tag{2.5}$$

$$R(\xi,X)Y = (\beta^2 + \xi\beta)[\eta(Y)X - g(X,Y)\xi], \tag{2.6}$$

$$\eta(R(X,Y)Z) = \beta^2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] - (X\beta)[g(Y,Z) - \eta(Y)\eta(Z)] + (Y\beta)[g(X,Z) - \eta(X)\eta(Z)], \tag{2.7}$$

$$S(X,\xi) = -(2n\beta^2 + \xi\beta)\eta(X) - (2n-1)(X\beta), \tag{2.8}$$

$$S(\xi,\xi) = -2n(\beta^2 + \xi\beta), \tag{2.9}$$

$$Q\xi = -(2n\beta^2 + \xi\beta)\xi - (2n-1)grad\beta, \tag{2.10}$$

for any vector field X, Y, Z on M, where R is the Riemannian curvature tensor, S is the Ricci tensor of type (0,2) and Q is the Ricci operator.

3 Parallel symmetric second order tensors and Ricci solitons in β-Kenmotsu manifolds

Let h denote a (0,2) type symmetric tensor field which is parallel with respect to ∇ that is $\nabla h = 0$. Then it follows that ([28],[24]):

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{3.1}$$

which gives

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0. \tag{3.2}$$

Taking $Z = W = \xi$ in (3.2) and using (2.5), we have

$$\beta^2[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)] - (X\beta)[h(Y,\xi) - \eta(Y)h(\xi,\xi)] + (Y\beta)[h(X,\xi) - \eta(X)h(\xi,\xi)] = 0. \tag{3.3}$$

With $X = \xi$ in (3.3) and by the symmetry of h, we have

$$(\beta^2 - \xi\beta)[\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{3.4}$$

Since $\beta^2 - \xi\beta \neq 0$, so by (3.4), we have

$$h(Y,\xi) = \eta(Y)h(\xi,\xi). \tag{3.5}$$

Differentiating (3.5) covariantly with respect to X, we have

$$\begin{aligned}(\nabla_X h)(Y,\xi) + h(\nabla_X Y,\xi) + h(Y,\nabla_X\xi)&\\ &= [(\nabla_X\eta)(Y) + \eta(\nabla_X Y)]h(\xi,\xi)\\ &+ \eta(Y)[(\nabla_X h)(\xi,\xi) + 2h(\nabla_X\xi,\xi)].\end{aligned} \tag{3.6}$$

By using (2.4), (3.5) and the parallel condition $\nabla h = 0$ in (3.6), we have

$$h(X,Y) = g(X,Y)h(\xi,\xi). \tag{3.7}$$

which implies that the parallelism of h gives $h(\xi,\xi)$ is a constant, via (3.5). So we have the following theorem.

Theorem 3.1. *A symmetric parallel second order covariant tensor in β-Kenmotsu manifold is a constant multiple of the metric tensor.*

Corollary 3.2. *A locally Ricci symmetric* ($\nabla S = 0$) *β-Kenmotsu manifold is an Einstein manifold.*

Remark 3.1. The following statements for β-Kenmotsu manifold are equivalent:

(i) Einstein,
(ii) locally Ricci symmetric,
(iii) Ricci semi-symmetric, that is $R\cdot S = 0$.

The implication $(i) \to (ii) \to (iii)$ is trivial. Now we prove that the implication $(iii) \to (i)$ in the more general frame work of β-Kenmotsu manifold. Since $R\cdot S = 0$, means exactly (3.2) with h replaced by S, that is

$$(R(X,Y)\cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V). \tag{3.8}$$

Taking $R\cdot S = 0$ and putting $X = \xi$ in (3.8), we have

$$S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \tag{3.9}$$

In view of (2.6) and $\beta^2 + \xi\beta \neq 0$, the above equation becomes

$$\begin{aligned}&\{\eta(U)S(Y,V) - g(Y,U)S(\xi,V)\}\\ &+ \{\eta(V)S(U,Y) - g(Y,V)S(U,\xi)\} = 0.\end{aligned} \tag{3.10}$$

Putting $U = \xi$ in (3.10) and by using (2.1), (2.8) and (2.9), we obtain

$$\begin{aligned}S(Y,V) = &- 2n(\beta^2 + \xi\beta)g(Y,V) + (2n-1)(Y\beta)\eta(V)\\ &- (2n-1)(V\beta)\eta(Y).\end{aligned} \tag{3.11}$$

If $\omega(X) = g(X, \rho) = X\beta = g(grad\beta, X)$ for all X, then (3.11) yields

$$\begin{aligned} S(Y, V) = &- 2n(\beta^2 + \xi\beta)g(Y, V) + (2n-1)\{\eta(V)\omega(Y) \\ &- \eta(Y)\omega(V)\}. \end{aligned} \tag{3.12}$$

From (3.12), it follows that a Ricci semi-symmetric β-Kenmotsu manifold is an Einstein manifold if and only if

$$\eta(V)\omega(Y) = \eta(Y)\omega(V), \tag{3.13}$$

that is, the vector field ξ and $\rho = grad\beta$ are parallel.
This leads to the following theorem.

Theorem 3.3. *A Ricci semi-symmetric β-Kenmotsu manifold (M, g) is an Einstein manifold if and only if the structure vector field ξ and the scalar potential of the structure function β are parallel.*

Corollary 3.4. *If on a β-Kenmotsu manifold the tensor field $(\mathcal{L}_V g + 2S)$ is ∇-parallel, then (g, V, λ) gives a Ricci soliton.*

Proof. A Ricci soliton in β-Kenmotsu manifold defined by (1.1) which gives $(\mathcal{L}_V g + 2S)$ is parallel. By theorem (3.1) it is clear that a symmetric parallel (0,2) tensor in β-Kenmotsu manifold is a constant multiple of metric tensor. Hence $(\mathcal{L}_V g + 2S)$ is a constant multiple of metric tensor g that is $(\mathcal{L}_V g + 2S)(X, Y) = g(X, Y)h(\xi, \xi)$, where $h(\xi, \xi)$ is a non-zero constant. It is the application of the theorem (3.1) to Ricci soliton. □

Theorem 3.5. *If a metric g in β-Kenmotsu manifold is a Ricci soliton with $V = \xi$, then it is η-Einstein.*

Proof. Taking $V = \xi$ in (1.2), we obtain

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{3.14}$$

Substituting

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi), \tag{3.15}$$

in (3.14) and by use of (2.4), we obtain

$$S(X, Y) = -(\beta + \lambda)g(X, Y) + \beta\eta(X)\eta(Y),$$

hence the result. □

Theorem 3.6. *A Ricci soliton (g, ξ, λ) in an n-dimensional β-Kenmotsu manifold can not be steady but is expanding.*

Proof. In linear algebra either the vector field $V \in \text{Span}\ \xi$ or $V \perp \xi$. But the second case $V \perp \xi$ seems to be complex to analyze in practice. For this reason we investigate for the case $V = \xi$.
By a simple computation of $(\mathcal{L}_V g + 2S)$, we obtain

$$(\mathcal{L}_\xi g)(X, Y) = 0. \tag{3.16}$$

From (1.2), we have

$$h(X, Y) = -2\lambda g(X, Y).$$

On putting $X = Y = \xi$, we obtain from above relation as

$$h(\xi, \xi) = -2\lambda, \tag{3.17}$$

where

$$h(\xi, \xi) = (\mathcal{L}_\xi g)(\xi, \xi) + 2S(\xi, \xi). \tag{3.18}$$

Using (2.9) and (3.16) we get from above

$$h(\xi, \xi) = -4n(\beta^2 + \xi\beta). \tag{3.19}$$

By virtue of (3.17) and (3.19), it follows that

$$\lambda = 2n(\beta^2 + \xi\beta).$$

Since β is some non-zero function, we have $\lambda \neq 0$ and so Ricci soliton in an n-dimension β-Kenmotsu manifold can not be steady but is expending because $\lambda > 0$. □

Theorem 3.7. *If an n-dimensional β-Kenmotsu manifold is η-Einstein then the Ricci soliton in β-Kenmotsu manifold that is (g, ξ, λ), where $\lambda = 2n\beta^2 + \xi\beta$ with varying scalar curvature can not be steady but it is expending.*

Proof. The proof consists of three parts.

(i) We prove that β-Kenmotsu manifold is η-Einstein,

(ii) We prove that the Ricci soliton in β-Kenmotsu manifold is consisting of varying scalar curvature,

(iii) We prove that the Ricci soliton in β-Kenmotsu manifold is expanding.

First we prove that the β-Kenmotsu manifold is η-Einstein: The metric g is called η-Einstein if there exists two real functions a and b such that the Ricci tensor of g is given by the general equation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{3.20}$$

Let $e_i, i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (3.20) and taking summation over i, we get

$$r = na + b. \tag{3.21}$$

Again putting $X = Y = \xi$ in (3.20) then by use of (2.9), we have

$$a + b = -(2n\beta^2 + \xi\beta). \tag{3.22}$$

Then from (3.21) and (3.22), we have

$$a = \frac{r + (2n\beta^2 + \xi\beta)}{n-1}, \qquad b = -\left[\frac{r + n(2n\beta^2 + \xi\beta)}{n-1}\right]. \tag{3.23}$$

Substituting the values of a and b in (3.20), we have

$$\begin{aligned} S(X,Y) = &\left[\frac{r + (2n\beta^2 + \xi\beta)}{n-1}\right] g(X,Y) \\ &- \left[\frac{r + n(2n\beta^2 + \xi\beta)}{n-1}\right] \eta(X)\eta(Y), \end{aligned} \tag{3.24}$$

the above equation shows that β-Kenmotsu manifold is an η-Einstein manifold.

Now, we have to show that the scalar curvature r is not a constant and it is varying. For an n-dimensional β-Kenmotsu manifold the symmetric parallel covariant tensor $h(X,Y)$ of type $(0,2)$ is given by

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{3.25}$$

Using (3.16) and (3.24) in (3.25), we have

$$\begin{aligned} h(X,Y) = &\, 2\left[\beta + \frac{r + (2n\beta^2 + \xi\beta)}{n-1}\right] g(X,Y) \\ &- 2\left[\beta + \frac{r + n(2n\beta^2 + \xi\beta)}{n-1}\right] \eta(X)\eta(Y). \end{aligned} \tag{3.26}$$

Differentiating (3.26) covariantly with respect to Z, we have

$$\begin{aligned}(\nabla_Z h)(X,Y) &= 2\left[Z\beta + \frac{(\nabla_Z r) + 4nZ\beta + Z(\xi\beta)}{n-1}\right] g(X,Y) \\ &- 2\left[Z\beta + \frac{(\nabla_Z r) + n(4nZ\beta + Z(\xi\beta))}{n-1}\right]\eta(X)\eta(Y) \\ &- 2\beta\left[\beta + \frac{r + n(2n\beta^2 + \xi\beta)}{n-1}\right]\{g(Z,X)\eta(Y) + g(Z,Y)\eta(X)\}.\end{aligned} \tag{3.27}$$

By substituting $Z = \xi$ and $X = Y \in (Span\ \xi)^{\perp}$ in (3.27) and using $\nabla h = 0$, we have

$$\nabla_\xi r = -(n-1)\nabla_\xi\beta - [4n\nabla_\xi\beta + \nabla_\xi(\xi\beta)]\,. \tag{3.28}$$

On integrating (3.28), we have

$$r = -(5n-1)\beta - \xi\beta + c, \tag{3.29}$$

where c is some integral constant. Thus from (3.29) we have r is a varying scalar curvature.

Finally, we have to check the nature of the soliton that is Ricci soliton in β-Kenmotsu manifold.

From (1.2), we have $h(X,Y) = -2\lambda g(X,Y)$, then putting $X = Y = \xi$, we have

$$h(\xi,\xi) = -2\lambda. \tag{3.30}$$

On putting $X = Y = \xi$ in (3.26), we have

$$h(\xi,\xi) = -2(2n\beta^2 + \xi\beta). \tag{3.31}$$

Equating (3.30) and (3.31), we have

$$\lambda = 2n\beta^2 + \xi\beta. \tag{3.32}$$

Since, $\lambda \neq 0$ because β is smooth function and $\lambda > 0$, that is the Ricci soliton in β-Kenmotsu manifold is expending. □

4 Ricci solitons in 3-Dimensional β-Kenmotsu manifold

Theorem 4.1. *In a Ricci soliton (g,ξ,λ) where $\lambda = 6\beta^2 + \frac{1}{2}\xi\beta$ of 3-dimensional β-Kenmotsu manifold with varying scalar curvature can not be steady but it is expending.*

Proof. The proof consists of three parts.

(i) We prove that the Riemannian curvature tensor of 3-dimensional β-Kenmotsu manifold is η-Einstein manifold,

(ii) We prove that the Ricci soliton in 3-dimensional β-Kenmotsu manifold is consisting of varying scalar curvature,

(iii) We prove that the Ricci soliton in a 3-dimensional β-Kenmotsu manifold is expending.

The Riemannian curvature tensor of 3-dimensional β-Kenmotsu manifold is given by

$$\begin{aligned} R(X,Y,Z) &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &- S(X,Z)Y - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{4.1}$$

Putting $Z = \xi$ in (4.1) and by using (2.5) and (2.8), we have

$$\begin{aligned} -\beta^2\{\eta(Y)X - \eta(X)Y\} &+ (X\beta)\{Y - \eta(Y)\xi\} - (Y\beta)\{X - \eta(X)\xi\} \\ &= \eta(Y)QX - \eta(X)QY - (6\beta^2 + \xi\beta)\eta(Y)X - 5(Y\beta)X \\ &+ (6\beta^2 + \xi\beta)\eta(X)Y + 5(X\beta)Y - \frac{r}{2}\{\eta(Y)X - \eta(X)Y\}. \end{aligned} \tag{4.2}$$

Again putting $Y = \xi$ in (4.2) and by using (2.1) and (2.10), we have

$$\begin{aligned} QX &= \left(\frac{r}{2} + 5\beta^2 + 5\xi\beta\right)X - \left(\frac{r}{2} + 11\beta^2 + \xi\beta\right)\xi \\ &+ 5(grad\ \beta)\eta(X) - 5(X\beta)\xi. \end{aligned} \tag{4.3}$$

By taking inner product of (4.3) with Y, we get

$$\begin{aligned} S(X,Y) &= \left(\frac{r}{2} + 5\beta^2 + 5\xi\beta\right)g(X,Y) - \left(\frac{r}{2} + 11\beta^2 + \xi\beta\right)\eta(X)\eta(Y) \\ &+ 5(Y\beta)\eta(X) - 5(X\beta)\eta(Y). \end{aligned} \tag{4.4}$$

Interchanging X and Y in (4.4), we have

$$\begin{aligned} S(Y,X) &= \left(\frac{r}{2} + 5\beta^2 + 5\xi\beta\right)g(Y,X) - \left(\frac{r}{2} + 11\beta^2 + \xi\beta\right)\eta(Y)\eta(X) \\ &+ 5(X\beta)\eta(Y) - 5(Y\beta)\eta(X). \end{aligned} \tag{4.5}$$

Adding (4.4) and (4.5), we have

$$\begin{aligned} S(X,Y) &= \left(\frac{r}{2} + 5\beta^2 + 5\xi\beta\right)g(X,Y) \\ &- \left(\frac{r}{2} + 11\beta^2 + \xi\beta\right)\eta(X)\eta(Y). \end{aligned} \tag{4.6}$$

This shows that a 3-dimensional β-Kenmotsu manifold is η-Einstein manifold.

Now, we would like to show that the scalar curvature r is not a constant that is r is varying.

For a 3-dimensional β-Kenmotsu manifold the symmetric parallel covariant tensor $h(X,Y)$ of type $(0,2)$ is given by

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{4.7}$$

By using (3.16) and (4.6) in (4.7), we have

$$h(X,Y) = (r + 10\beta^2 + 10\xi\beta)g(X,Y) - (r + 22\beta^2 + 2\xi\beta)\eta(X)\eta(Y). \tag{4.8}$$

Differentiating the above equation covariantly with respect to Z, we have

$$\begin{aligned}(\nabla_Z h)(X,Y) &= \{\nabla_Z r + 20\beta(Z\beta) + 10Z(\xi\beta)\}g(X,Y)\\ &- \{\nabla_Z r + 44\beta(Z\beta) + 2Z(\xi\beta)\}\eta(X)\eta(Y)\\ &- (r + 22\beta^2 + 2\xi\beta)\{(\nabla_Z\eta)(X)\eta(Y) - \eta(X)(\nabla_Z\eta)(Y)\}.\end{aligned} \tag{4.9}$$

Substituting $Z = \xi$ and $X = Y \in (Span\ \xi)^\perp$ in (4.9) and by virtue of $\nabla h = 0$, we have

$$\{\nabla_\xi r + 10\nabla_\xi(\beta^2) + 10\nabla_\xi(\xi\beta)\} = 0. \tag{4.10}$$

On integrating (4.10), we have

$$r = -10(\beta^2 + \xi\beta) + c. \tag{4.11}$$

where c is integral constant. Thus from (4.11), we have r a variable scalar curvature.

Finally, we have to check the nature of the Ricci soliton (g,ξ,η) in 3-dimensional β-Kenmotsu manifold.

From (1.2), we have

$$h(X,Y) = -2\lambda g(X,Y). \tag{4.12}$$

On putting $X = Y = \xi$ in (4.12), we have

$$h(\xi,\xi) = -2\lambda. \tag{4.13}$$

On taking $X = Y = \xi$ in (4.8), we have

$$h(\xi,\xi) = -12\beta^2 - \xi\beta. \tag{4.14}$$

Equating (4.13) and (4.16), we have

$$\lambda = 6\beta^2 + \frac{1}{2}\xi\beta. \tag{4.15}$$

Since from (4.15), $\lambda \neq 0$ and $\lambda > 0$, therefore Ricci soliton (g,ξ,η) in 3-dimensional β-Kenmotsu manifold is expending. □

Example 4.1. Let $M = \{(x, y, z) \in R^3 : z \neq 0\}$, where (x, y, z) are standard co-ordinate in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent vector fields given by

$$E_1 = z^2 \frac{\partial}{\partial x}, \qquad E_2 = z^2 \frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0,$$
$$g(E_1, E_1) = g(E_2, E_2) = g(E_2, E_3) = 1.$$

Let η be a 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$ and φ be the $(1, 1)$-tensor field defined by

$$\varphi E_1 = -E_2, \qquad \varphi E_2 = E_1 \quad \textit{and} \quad \varphi E_3 = 0.$$

Then using the linearity of φ on g, we have

$$\eta(E_3) = 1, \qquad \varphi^2 U = -U + \eta(U)E_3,$$

and $$g(\varphi U, \varphi W) = g(U, W) - \eta(U)\eta(W),$$
for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.
Let ∇ be the Riemannian connection of g, then we have

$$[E_1, E_2] = 0, \qquad [E_1, E_3] = -\frac{2}{z}E_1 \quad \textit{and} \quad [E_2, E_3] = -\frac{2}{z}E_2.$$

Koszul formula is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

By using the Koszul formula for the Riemannian metric g, we can easily calculate

$$\begin{aligned} &\nabla_{E_1} E_1 = \frac{2}{z}E_3, && \nabla_{E_1} E_2 = 0, && \nabla_{E_1} E_3 = -\frac{2}{z}E_1, \\ &\nabla_{E_2} E_1 = 0, && \nabla_{E_2} E_2 = \frac{2}{z}E_3, && \nabla_{E_2} E_3 = -\frac{2}{z}E_2, \\ &\nabla_{E_3} E_1 = 0, && \nabla_{E_3} E_2 = 0, && \nabla_{E_3} E_3 = 0. \end{aligned}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is β-Kenmotsu structure on M and satisfy

$$(\nabla_X \varphi)Y = \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X], \qquad \nabla_X \xi = \beta[X - \eta(X)\xi], \tag{4.16}$$

where $\beta = -\frac{2}{z}$. Hence structure (ϕ, ξ, η, g) defines a β-Kenmotsu structure. Thus M equipped with β-Kenmotsu structure is a β-Kenmotsu manifold. The tangent vector X and Y on M are expressed as linear combination of E_1, E_2, E_3, that is

$$X = a_1E_1 + a_2E_2 + a_3E_3,$$
$$Y = b_1E_1 + b_2E_2 + b_3E_3,$$

where a_i and b_i, $(i = 1, 2, 3)$ are scalars.
Using $\beta = -\frac{2}{z}$ in (4.11), we have

$$r = -\frac{60}{z^2} + c,$$

which shows that, the scalar curvature r is not constant.
Using $\beta = -\frac{2}{z}$ in (4.15), we have

$$\lambda = \frac{25}{z^2},$$

this implies that $\lambda > 0$, that is the Ricci soliton in 3-dimensional β-Kenmotsu manifold is expending.

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