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SEMI GENERALIZED φ -RECURRENT TRANS-SASAKIAN MANIFOLDS

Jagannath Chowdhury*, Rajesh Kumar and Jay P. Singh

Abstract. In this paper we studied semi generalized φ -recurrent and concircular φ -recurrent Trans-Sasakian manifolds.

keywords: Trans-Sasakian manifold, semi generalized φ -recurrent, η -Einstein, η -parallel, constant curvature.

1. Introduction

In 1985, Oubina [3] introduced a class of almost contact metric manifolds known as trans-Sasakian manifolds. This class contains α -Sasakian, β -Kenmotsu and co-symplectic manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure if the product manifold $M \times R$ belongs to the class W_4 , a class of Hermitian manifolds which are closely related to a locally conformal Kähler manifolds. Trans-Sasakian manifolds were studied extensively by Marrero [10], Tripathi [12], De et al. ([5], [7], [8]) and others. Tripathi [12] proved that trans-Sasakian manifolds are always generalized quasi-Sasakian. De et al. and Bagewadi et al. ([4], [1]) have obtained results on the conservativeness of Projective, Pseudo projective, Conformal concircular, Quasi conformal curvature tensors on k -contact, Kenmotsu and trans-Sasakian manifolds. De et al. [5] generalized the notion of locally φ -symmetric and introduced the notion of φ -recurrent Sasakian manifolds. The author Nagaraja [11] introduced the notion of φ -recurrent trans-Sasakian manifolds. In the present paper we study generalized φ -recurrent and concircular φ -recurrent Trans-Sasakian manifolds.

2. Preliminaries

Let M be a n dimensional almost contact metric manifold [6] with an almost contact metric structure (φ, ξ, η, g) where φ, ξ, η are tensor fields on M of types $(1,1), (1,0),$

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(0,1) respectively and g is the Riemannian metric on M such that

$$(2.1) \quad \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0.$$

The Riemannian metric g on M satisfies the condition

$$(2.2) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.3) \quad g(X, \varphi Y) = -g(\varphi X, Y), \quad g(X, \xi) = \eta(X),$$

for all $X, Y \in TM$. An almost contact metric structure (φ, ξ, η, g) in M is called a trans-Sasakian structure [2] if the product manifold $(M \times R, J, G)$ belongs to the class W_4 where J is the complex structure on $(M \times R)$ defined by

$$(2.4) \quad J(X, \lambda \frac{d}{dt}) = (\varphi X - \lambda \xi, \eta(X) \frac{d}{dt}),$$

for all vector fields X on M and smooth functions λ on $(M \times R)$ and G is the product metric on $(M \times R)$. This may be expressed by the following condition [7]

$$(2.5) \quad (\nabla_X \varphi)(Y) = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X),$$

where α and β are smooth functions on M . From (2.5), we have

$$(2.6) \quad \nabla_X \xi = -\alpha(\varphi X) + \beta(X, \eta(X)\xi)$$

and

$$(2.7) \quad (\nabla_X \eta)(Y) = -\alpha g(\varphi X, Y) + \beta(\varphi X, \varphi Y).$$

In an n -dimensional trans-Sasakian manifold, from (2.5) (2.6) (2.7), we can derive [7]

$$(2.8) \quad \begin{aligned} R(X, Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) \\ &+ 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) - (X\alpha)\varphi Y \\ &+ (Y\alpha)\varphi X - (X\beta)\varphi^2 Y + (Y\beta)\varphi^2 X, \end{aligned}$$

$$(2.9) \quad S(X, \xi) = \{(n-1)(\alpha^2 - \beta^2) - \xi\beta\} \eta(X) - (n-2)(X\beta) - (\varphi X)\alpha$$

and

$$(2.10) \quad Q\xi = \{(n-1)(\alpha^2 - \beta^2) - \xi\beta\} \xi - (n-2)\text{grad}\beta + \varphi(\text{grad}\alpha),$$

where R is the curvature tensor, S is the Ricci-tensor and r is the scalar curvature. Also

$$(2.11) \quad g(QX, Y) = S(X, Y),$$

where Q being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S .

When

$$(2.12) \quad \varphi(\text{grad}\alpha) = (n-2)\text{grad}\beta,$$

then (2.9) and (2.10) reduces to

$$(2.13) \quad S(X, \xi) = (n-1)(\alpha^2 - \beta^2)\eta(X),$$

and

$$(2.14) \quad Q\xi = (n-1)(\alpha^2 - \beta^2)\xi.$$

A Sasakian manifold is said to be a φ -recurrent manifold if there exists a non zero 1-form A such that

$$(2.15) \quad \varphi^2((\nabla_W R)(X, Y)Z) = A(X)R(Y, Z)W$$

for all vector fields X, Y, Z, W orthogonal to ξ .

Further we have

$$(2.16) \quad 2\alpha\beta + \xi\alpha = 0.$$

3. Semi-generalized φ -recurrent trans Sasakian Manifolds

Definition 3.1. A trans-Sasakian manifold (M^n, g) is called semi-generalized φ -recurrent if its curvature tensor R satisfies the condition

$$(3.1) \quad \varphi^2\left((\nabla_W R)(X, Y)Z\right) = A(W)R(X, Y)Z + B(W)g(Y, Z)X,$$

for all $X, Y, Z, W \in TM$, where A and B are two 1-forms, B is non zero and these are defined by

$$(3.2) \quad A(W) = g(W, \rho_1), \quad B(W) = g(W, \rho_2),$$

where ρ_1, ρ_2 being the vector fields associated to the 1-form A, B respectively.

Theorem 3.1. A semi-generalized φ -recurrent trans-Sasakian manifold (M^n, g) satisfying $\varphi(\text{grad}\alpha) = (n-2)\text{grad}\beta$, is an Einstein manifold and more over the 1-forms A and B are related as

$$(n-1)(\alpha^2 - \beta^2)A(W) + nB(W) = 0.$$

Proof: Let us consider a semi-generalized φ -recurrent trans-Sasakian manifold. Then by virtue of (2.1) and (3.1) we have

$$(3.3) \quad \begin{aligned} & - (\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi \\ & = A(W)R(X, Y)Z + B(W)g(Y, Z)X, \end{aligned}$$

from which we get

$$(3.4) \quad \begin{aligned} & - g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) \\ & = A(W)R(X, Y, Z, U) + B(W)g(Y, Z)g(X, U). \end{aligned}$$

Let $\{e_i\}$, $i = 1, 2, \dots, n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.4) and taking summation over i , $1 \leq i \leq n$, we get

$$(3.5) \quad \begin{aligned} -(\nabla_W S)(Y, Z) & + \sum_{i=1}^n \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) \\ & = A(W)S(Y, Z) + nB(W)g(Y, Z). \end{aligned}$$

The second term of (3.5) by putting $Z = \xi$ takes the form $g((\nabla_W R)(e_i, Y)\xi, e_i)$.

Consider

$$(3.6) \quad \begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) & = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(\nabla_W e_i, Y)\xi, \xi) \\ & - g(R(e_i, \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi) \end{aligned}$$

at $p \in M$. Since $\{e_i\}$ is an orthonormal basis, so $\nabla_X e_i = 0$ at p . Using (2.1), (2.3) and (2.8), we have

$$(3.7) \quad \begin{aligned} g(R(e_i, \nabla_W Y)\xi, \xi) & = g\left((\alpha^2 - \beta^2)(\eta(\nabla_W Y)e_i - \eta(e_i)(\nabla_W Y)) \right. \\ & + 2\alpha\beta(\eta(\nabla_W Y)\varphi e_i - \eta(e_i)\varphi(\nabla_W Y)) \\ & + (\nabla_W Y\alpha)\varphi e_i - (e_i\alpha)\varphi(\nabla_W Y) \\ & \left. + (\nabla_W Y\beta)\varphi^2 e_i - (e_i\beta)\varphi^2(\nabla_W Y), \xi\right) \\ & = 0. \end{aligned}$$

Using (3.7) in (3.6), we obtain

$$(3.8) \quad g((\nabla_W R)(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).$$

Since $(\nabla_W g) = 0$, we have

$$g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0,$$

which implies

$$(3.9) \quad g(\nabla_W R(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).$$

Using (2.6) in (3.9), we get

$$\begin{aligned}
 g((\nabla_W R)(e_i, Y)\xi, \xi) &= -g\left(R(e_i, Y)\xi, -\alpha\varphi W + \beta(W - \eta(W)\xi)\right) \\
 &\quad - g\left(R(e_i, Y)(-\alpha\varphi W + \beta(W - \eta(W)\xi)), \xi\right) \\
 &= \alpha g(R(e_i, Y)\xi, \varphi W) - \beta g(R(e_i, Y)\xi, W) \\
 &\quad + \beta \eta(W)g(R(e_i, Y)\xi, \xi) + \alpha g(R(e_i, Y)\varphi W, \xi) \\
 &\quad - \beta g(R(e_i, Y)W, \xi) + \beta \eta(W)g(R(e_i, Y)\xi, \xi) \\
 (3.10) \qquad \qquad \qquad &= 0.
 \end{aligned}$$

Replacing Z by ξ in (3.5) and using (2.3), (2.13) and (3.10) we have

$$(3.11) \quad (\nabla_W S)(Y, \xi) = -[(n-1)(\alpha^2 - \beta^2)A(W) + nB(W)]\eta(Y).$$

Now, we have

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).$$

Using (2.6) and (2.13) in the above relation, it follows that

$$\begin{aligned}
 (\nabla_W S)(Y, \xi) &= (n-1)(\alpha^2 - \beta^2)[- \alpha g(\varphi W, Y) + \beta g(\varphi Y, \varphi W)] \\
 (3.12) \quad &+ \alpha S(Y, \varphi W) - S(Y, \beta W) + (n-1)\beta(\alpha^2 - \beta^2)\eta(Y)\eta(W).
 \end{aligned}$$

By virtue of (2.2), we obtain from (3.12) that

$$\begin{aligned}
 (\nabla_W S)(Y, \xi) &= (n-1)(\alpha^2 - \beta^2)[- \alpha g(Y, \varphi W) + \beta g(Y, W)] \\
 (3.13) \quad &+ \alpha S(Y, \varphi W) - \beta S(Y, W).
 \end{aligned}$$

From (3.11) and (3.13), we have

$$\begin{aligned}
 (n-1)(\alpha^2 - \beta^2)[- \alpha g(Y, \varphi W) + \beta g(Y, W)] + \alpha S(Y, \varphi W) \\
 (3.14) \quad - \beta S(Y, W) &= -[(n-1)(\alpha^2 - \beta^2)A(W) + nB(W)]\eta(Y).
 \end{aligned}$$

Replacing $Y = \xi$ in (3.14) then using (2.1), (2.3), (2.12) and (2.13) we get

$$(3.15) \quad (n-1)(\alpha^2 - \beta^2)A(W) + nB(W) = 0.$$

Again replacing Y and W by φY and φW respectively in (3.14) and then using (2.1), (2.3), (2.11), (2.12) and (2.14), we obtain

$$(3.16) \quad S(Y, W) = (n-1)(\alpha^2 - \beta^2)g(Y, W)$$

and

$$S(\varphi Y, W) = (n-1)(\alpha^2 - \beta^2)g(\varphi Y, W).$$

Which proves the theorem.

4. Semi-generalized concircular φ -recurrent trans-Sasakian Manifolds

Definition 4.1. A trans-Sasakian manifold is said to be concircular φ -recurrent manifold if there exists a non zero 1-form A such that

$$(4.1) \quad \varphi^2((\nabla_W C)(X, Y)Z) = A(W)C(X, Y)Z,$$

for arbitrary vector fields X, Y, Z, W where C is a concircular curvature tensor given by

$$(4.2) \quad C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y],$$

where R is the curvature tensor and r is the scalar curvature.

Definition 4.2. A trans-Sasakian manifold is called a semi-generalized concircular φ -recurrent if its concircular curvature tensor C defined in (4.2) satisfies the condition

$$(4.3) \quad \varphi^2(\nabla_W C)(X, Y, Z) = A(W)C(X, Y, Z) + B(W)g(Y, Z)X,$$

where A and B are defined as (3.2).

Theorem 4.1. Let (M^n, g) be a semi-generalized concircular φ -recurrent trans-Sasakian manifold then

$$\left[(n-1)(\alpha^2 - \beta^2) - \frac{r}{n} \right] A(W) + nB(W) = 0.$$

Proof:

Let (M^n, g) be a semi-generalized concircular φ -recurrent trans-Sasakian manifold. Then by virtue of (2.1) and (4.3), we have

$$(4.4) \quad \begin{aligned} & - (\nabla_W C)(X, Y, Z) + \eta(\nabla_W C)(X, Y, Z))\xi \\ & = A(W)C(X, Y, Z) + B(W)g(Y, Z)X, \end{aligned}$$

from which it follows that

$$(4.5) \quad \begin{aligned} & - g((\nabla_W C)(X, Y, Z), U) + \eta((\nabla_W C)(X, Y, Z))\eta(U) \\ & = A(W)g(C(X, Y, Z), U) + B(W)g(Y, Z)g(X, U). \end{aligned}$$

Let $\{e_i\}, i = 1, 2, \dots, n$ be orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (4.5) and taking summation over i , $1 \leq i \leq n$, we get

$$(4.6) \quad \begin{aligned} & -(\nabla_W S)(X, U) + \frac{W(r)}{n}g(X, U) + (\nabla_W S)(X, \xi)\eta(U) \\ & - \frac{W(r)}{n}\eta(X)\eta(U) = A(W) \left[S(X, U) - \frac{r}{n}g(X, U) \right] \\ & + nB(W)g(X, U). \end{aligned}$$

Replacing U by ξ in (4.6) and using (2.1) and (2.13), we get

$$(4.7) \quad A(W) \left[(n-1)(\alpha^2 - \beta^2) - \frac{r}{n} \right] \eta(X) + nB(W)\eta(X) = 0.$$

Putting $X = \xi$ in (4.7), we have

$$(4.8) \quad \left[(n-1)(\alpha^2 - \beta^2) - \frac{r}{n} \right] A(W) + nB(W) = 0.$$

This completes the proof.

Theorem 4.2. *A semi-generalized concircular φ -recurrent trans-Sasakian manifold is an Einstein manifold, provided α and β are constants.*

Proof: Putting $X = U = e_i$ in (4.5) and taking summation over i , $1 \leq i \leq n$, we get

$$(4.9) \quad \begin{aligned} & - (\nabla_W S)(Y, Z) = - \sum_{i=1}^n g((\nabla_W R)(e_i, Y, Z), \xi) g(e_i, \xi) \\ & - \frac{W(r)}{n} g(Y, Z) + \frac{W(r)}{n(n-1)} \left[g(Y, Z) - \eta(Y)\eta(Z) \right] \\ & + A(W) \left[S(Y, Z) - \frac{r}{n} g(Y, Z) \right] + nB(W)g(Y, Z). \end{aligned}$$

Replacing Z by ξ in (4.9) and using (4.7), we have

$$(4.10) \quad (\nabla_W S)(Y, \xi) = \frac{W(r)}{n} \eta(Y).$$

Now, we have

$$(4.11) \quad (\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).$$

Using (2.6) and (2.9) in the above relation, it follows that

$$(4.12) \quad \begin{aligned} (\nabla_W S)(Y, \xi) &= (n-1)(\alpha^2 - \beta^2)[- \alpha g(\varphi W, Y) + \beta g(W, Y)] \\ &+ \alpha S(Y, \varphi W) - \beta S(Y, W). \end{aligned}$$

In view of (4.10) and (4.12)

$$(4.13) \quad \begin{aligned} & (n-1)(\alpha^2 - \beta^2)[- \alpha g(\varphi W, Y) + \beta g(W, Y)] \\ & + \alpha S(Y, \varphi W) - \beta S(Y, W) = \frac{W(r)}{n} \eta(Y). \end{aligned}$$

Replacing Y by φY in (4.13) and using (2.2), we get

$$(4.14) \quad \begin{aligned} & - \alpha S(\varphi Y, \varphi W) + \beta S(\varphi Y, W) \\ & = (n-1)(\alpha^2 - \beta^2)[\beta g(W, \varphi Y) - \alpha g(\varphi W, \varphi Y)]. \end{aligned}$$

Interchanging Y by W in (4.14) and by using the skew symmetry of φ , we obtain

$$(4.15) \quad \alpha S(\varphi W, \varphi Y) = (n-1)\alpha(\alpha^2 - \beta^2)g(\varphi W, \varphi Y).$$

By skew symmetry of φ and using (2.9), we obtain

$$\begin{aligned} S(\varphi W, \varphi Y) &= S(\varphi^2 W, Y) \\ &= S(W, Y) - (n-1)(\alpha^2 - \beta^2)\eta(W)\eta(Y). \end{aligned}$$

Substituting this in (4.15), we get

$$(4.16) \quad S(W, Y) = (n-1)(\alpha^2 - \beta^2)g(W, Y).$$

i.e M is an Einstein manifold. Hence the theorem is verified.

5. Conclusion

This paper is all about the study of geometrical properties of a semi-generalized ϕ -recurrent trans-Sasakian manifold. We prove that a semi-generalized ϕ -recurrent trans-Sasakian manifold is an Einstein manifold. It is proved that a semi-generalized concircular ϕ -recurrent trans-Sasakian manifold is also an Einstein manifold.

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